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Radiative Reaction and Quantum Mechanics

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According to the principles of quantum mechanics, the classical Lorentz–Dirac equations of the electron should follow from quantum electrodynamics in the classical limit. We show this is indeed true for the special case in which the charge does not radiate, provided the momentum operators in the Dirac theory are identified, in the classical limit, with the effective momenta of the Lorentz–Dirac equations.

The classical electron theory provides the following equations of motion for an electron in an external field of force F_μ^{ext} (Dirac, 1938; Tirapegui, 1978):

$$\frac{d}{ds}(m_0 + \Delta m)v_\mu = F_\mu^{\text{ext}} + \frac{2\alpha}{3}\{\dot{v}_\mu + \dot{v}_\rho \dot{v}^\rho v_\mu\} \tag{1}$$

According to the principles of quantum mechanics, these classical equations and their consequences should be consistent with and, in the limit of large quantum numbers, should follow from quantum mechanics. The clarification of this important and long-standing problem remains unsolved (Rohrlich, 1963). It is well known that the JWKB method applied to the Dirac equation yields the equations (1) without the radiation reaction terms (Rohrlich, 1963; Rubinow and Keller, 1963). Pilkuhn (1979) presents a derivation of the Dirac equation from quantum electrodynamics. This derivation, which is based on the Bethe–Salpeter equations, is clearly approximative, since it yieds the Dirac equation, and the Dirac equation does not give the exact equations (1) in the classical limit. A resolution of this problem will follow from a much deeper understanding of quantum electrodynamics and its classical limit (Bialynicki-Birula, 1971). However, is it possible that, at least in some special cases, this discrepancy between

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the classical and quantum theories may be resolved? We wish to show that it is indeed so.

As our starting point we consider the description of a classical charged particle given in Browne (1970). In that paper an effective motion of a charged particle is considered; it is distinguished from the motion as described by equation (1) in that the charge performs a *classical Zitterbewegung* about the effective motion. First, define the *effective electron momenta* (Browne, 1969, 1970)

$$P_\alpha = (m_0 + \Delta m)v_\alpha - \kappa \dot{v}_\alpha \tag{2}$$

and an *effective mass*

$$M^2 = P^\alpha P_\alpha = m^2 + \kappa^2 I \qquad (m = m_0 + \Delta m) \tag{3}$$

where $\kappa = 2\alpha/3$ and

$$I = \dot{v}_\mu \dot{v}^\mu = \gamma^6[\mathbf{a}^2 - (\mathbf{v} \times \mathbf{a})^2] \tag{4}$$

$\gamma = (1 - v^2/c^2)^{-1/2}$, and $\mathbf{a}$ and $\mathbf{v}$ are the three-acceleration and three-velocity of the charge, respectively. In obtaining equation (3) we have used (Dirac, 1938)

$$v_\mu v^\mu = 1, \qquad v_\mu \dot{v}^\mu = 0 \tag{5}$$

Expressing equation (1) in terms of the P_α, we obtain (Browne, 1970)

$$\dot{P}_\alpha = F_\alpha^{\text{ext}} + \kappa I v_\alpha \tag{6}$$

We claim that m^2 is a conserved quantity. To see this, dot both sides of equation (1) with v_μ to obtain

$$\frac{d}{ds}\left(\frac{1}{2} m v^2 - \kappa v^\mu \dot{v}_\mu\right) = 0 \tag{7}$$

Now use equation (5) to conclude

$$\frac{d}{ds}(m^2) = 0 \tag{8}$$

This result contradicts equation (15) of Browne (1970).[2]

{*Definitions and Notation.* The units we choose are such that $\hbar = c = 1$, and $\alpha = e^2$. Here $\Delta m = e^2/a$ is the electromagnetic mass. [a has various physical interpretations, such as that of a regularization parameter (Tirapegui, 1978) and also as the *classical electron radius* (Dirac, 1938;

[2]Browne's assumption [his equation (13)] with our equation (8) implies $I = 0$. This is the case we consider below.

Rohrlich, 1965)]. $F_\mu^{\text{ext}} = eF_{\mu\nu}^{\text{ext}} v^\nu$, where $F_{\mu\nu}^{\text{ext}}$ is the external electromagnetic field. Gamma matrices, metric, etc., are those of Bjorken and Drell (1964). The Einstein summation convention is used throughout. $\sigma_{\mu\nu} = (i/2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ are relativistic generalizations of the Pauli matrices, i.e., $\sigma_{12} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$, etc. The components of the spin angular momentum tensor are $S_{\mu\nu} = (i/2)\sigma_{\mu\nu}$. They are the generators of the spin representation of the Lorentz group, $SO_0(1, 3)$ (Cartan, 1966). In going over to quantum mechanics we have $\cdot = d/ds = \frac{1}{2}\gamma_0\, d/dt$, and for any observable O,

$$\frac{dO}{dt} = i[H, O] + \frac{\partial O}{\partial t} \tag{9}$$

where $H = \boldsymbol{\alpha} \cdot \mathbf{P} + \beta m$ is the Dirac Hamiltonian for a spin 1/2 system having mass m. Also, $F_\alpha^{\text{ext}} \to (e/2)F_{\alpha\nu}^{\text{ext}}\gamma^\nu$ and $v_\alpha \to \frac{1}{2}\gamma_\alpha$ in going over to quantum mechanics.}

Let us now consider the special case of vanishing I. For circular motion, $\mathbf{a} \times \mathbf{v} = 0$, so that $I = 0$ implies $\mathbf{a} = 0$ by equation (4). However, this need not be true if $\mathbf{a}$ and $\mathbf{v}$ are not perpendicular to one another. For $I = 0$, (6) becomes

$$\dot{P}_\alpha = F_\alpha^{\text{ext}} \tag{10}$$

which equations are just the Lorentz force equations, i.e., they are the Lorentz–Dirac equations (1) excluding the radiation reaction terms. We will shortly consider a relativistic and quantum mechanical spin-1/2 system having real mass m and having momenta given by a quantum mechanical generalization of (2). If we can show that equations (10) follow in the classical limit from the equation of our quantum mechanical system, then we will have provided a resolution of the above-mentioned discrepancy between the classical and quantum theories, at least in cases for which I vanishes. We now proceed to show this.

In going over to quantum mechanics we consider the Dirac equation for a reall mass, spin-1/2 system

$$\gamma^\alpha P_\alpha \psi = m\psi \tag{11}$$

where

$$P_\alpha \equiv i\partial_\alpha - eA_\alpha \tag{12}$$

are the operators for the effective momenta of the charge. We insist that there be operator analogs of equations (2), and that they be

$$P_\alpha = \mathring{P}_\alpha + \frac{1}{2}\Delta m\gamma_\alpha - \frac{\kappa}{2}\dot{\gamma}_\alpha \tag{13}$$

acting on any solution of (11).

The *bare momenta* $\mathring{P}_\alpha$ will be determined from consistency between equation (11) and these equations. For this we consider the operator identity, which is valid acting on arbitrary solutions of the Dirac equation (Moylan, 1985)

$$P_\alpha = m\gamma_\alpha - i\dot{\gamma}_\alpha \tag{14}$$

Using this, we eliminate $\dot{\gamma}_\alpha$ from (13), and obtain for the bare momenta

$$\mathring{P}_\alpha = \left(1 + \frac{i\kappa}{2}\right)P_\alpha - \frac{\Delta m}{2}\gamma_\mu - \frac{i\kappa m}{2}\gamma_\alpha \tag{15}$$

If we let

$$\frac{\Delta m}{m} = -i\frac{2\alpha}{3} \tag{16}$$

then we obtain

$$\mathring{P}_\alpha = \left(1 + \frac{i\kappa}{2}\right)P_\alpha \tag{17}$$

Equation (16) implies the classically untenable and puzzling notion of an imaginary electromagnetic mass. Of course, since the electromagnetic mass is not a directly observed quantity, the value of Δm obtained from (16) does not pose any theoretical difficulty.

Equation (17) represents the effect on the momenta of a change of scale

$$x_\mu \to x'_\mu = \lambda^{-1}x_\mu, \qquad \lambda = \frac{1}{1 + \Delta m/m} \tag{18}$$

This conformal transformation [or more precisely, this element of the complexification, $SL(4, \mathbb{C})$, of the conformal group, $SU(2, 2)$] induces an action on spinor valued functions over complexified Minkowski space. Explicitly (Paneitz *et al.*, 1987)

$$U^w(g){:}\quad \mathring{\psi}(x) \to \psi(x) = [U^w(g)\mathring{\psi}](x) = S(\lambda)\mathring{\psi}(\lambda^{-1}x) \tag{19}$$

Under this transformation the Dirac equation (11) becomes (Paneitz *et al.*, 1987)

$$\mathring{\gamma}_\mu \mathring{P}^\mu \mathring{\psi}(x') = \left(m_0 + \frac{1}{2}\Delta m\right)\mathring{\psi}(x') \tag{20}$$

We have thus demonstrated the equivalence of (11) under changes of scale with the original Dirac equation of the theory, up to a mass renormalization. The existence of a conformally invariant interaction Lagrangian is assured for the case $w = \frac{3}{2}$ (Segal, 1991).

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